

L^p -STABILITY ($1 \leq p \leq \infty$) OF MULTIVARIABLE NONLINEAR TIME-VARYING FEEDBACK SYSTEMS THAT ARE OPEN-LOOP UNSTABLE.

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Abstract

This paper considers a class of multivariable, nonlinear time-varying feedback systems with an unstable convolution subsystem as feedforward and a time-varying nonlinear gain as feedback. The impulse response of the convolution subsystem is the sum of i) a finite number of increasing exponentials multiplied by nonnegative powers of the time t , ii) a term that is absolutely integrable and iii) a infinite series of delayed impulses. The main result of the paper is theorem 1. It essentially states that i) if the unstable convolution subsystem can be stabilized by a constant feedback gain F and ii) if the incremental gain of the difference between the nonlinear gain function and F is sufficiently small, then the nonlinear system is L^p -stable for any $p \in [1, \infty]$; furthermore the solutions of the nonlinear system depend continuously on the inputs in any L^p -norm. The fixed point theorem is crucial in deriving the above theorem.

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1. Introduction. In the past few years the L^2 -stability [1], [2] the L^∞ -stability [3], [4] of certain classes of nonlinear and time-varying feedback systems have been extensively studied. Desoer and Wu [5], [6] obtained L^p -stability conditions for a broad class of linear time-invariant feedback systems whose open-loop impulse responses may include an integration and an infinite series of delayed impulses. They also obtained L^p -stability conditions for a related class of nonlinear time-varying systems in [7]. Recently Callier and Desoer [8], [9], [10] derived necessary and sufficient conditions for stability of a very broad class of linear time-invariant feedback systems whose open-loop impulse responses may include increasing exponentials multiplied by nonnegative powers of time and an infinite series of delayed impulses. These conditions imply L^p -stability for any $p \in [1, \infty]$, [6]. In this paper the loop transformation technique [12], the fixed point theorem [16], and a generalized version of some results of Callier, Desoer and Wu [10], [7] are used to derive the L^p -stability for a related class of nonlinear time-varying feedback systems which are open-loop unstable. The application of the fixed point theorem in L^p shows that the nonlinear feedback system has one and only one solution for any pair of inputs in L^p , that the solutions are continuously dependent on the inputs and that closed loop system is L^p -stable for any $p \in [1, \infty]$.

2. Notations. In this paper we shall encounter real numbers (elements of \mathbb{R}), vectors (in \mathbb{R}^n), matrices (in $\mathbb{R}^{n \times n}$), elements in function spaces and operators acting on elements of function spaces. Lower-case letters denote numbers or vectors, upper-case letters denote matrices. Bold-face letters (indicated by a tilde under the symbol) denote operators. The symbol $|\cdot|$ denotes both the magnitude of a number and the norm of a vector in \mathbb{R}^n or a matrix in $\mathbb{R}^{n \times n}$. In function spaces, we use the following norms: Let $x: \mathbb{R}_+ \rightarrow \mathbb{R}^n$, then by definition

$$\|x\|_p \triangleq \left[\int_0^\infty |x(t)|^p dt \right]^{1/p}, \quad 1 \leq p < \infty,$$

and, for $p = \infty$,

$$\|x\|_{\infty} \triangleq \text{ess sup}_{t \geq 0} |x(t)|.$$

The resulting normed spaces are denoted by L_n^p , $1 \leq p \leq \infty$. (If $n = 1$ (scalar case) we write L^p .) When the symbols $|\cdot|$ and $\|\cdot\|$ are applied to a matrix or a matrix-valued function or an operator acting on function spaces, they denote the induced operator norms. Note that in defining the L_n^p norms above we may use any vector norm in \mathbb{R}^n because all norms in \mathbb{R}^n are equivalent. Following Sandberg [11] and Zames [2], the space L_{ne}^p , the extension of L_n^p space, is defined as follows:

$$L_{ne}^p \triangleq \{x(\cdot) \mid \int_0^T |x(t)|^p dt < \infty, \forall T \in [0, \infty), 1 \leq p < \infty\}$$

and

$$L_{ne}^{\infty} \triangleq \{x(\cdot) \mid \text{ess sup}_{t \in [0, T]} |x(t)| < \infty, \forall T \in [0, \infty)\}.$$

Roughly speaking, if $x \in L_{ne}^{\infty}$, then x does not have a finite escape time.

In order to allow us to consider a larger class of linear subsystems whose impulse responses may include an infinite series of impulses, we introduce the Banach Algebra $\mathcal{A}^{n \times n}$ (see [6]). Let A be a distribution whose support is in $[0, \infty)$. We say that A is an element of $\mathcal{A}^{n \times n}$ if

$$A(t) = A_a(t) + \sum_{i=0}^{\infty} A_i \delta(t-t_i)$$

where $A_a: [0, \infty) \rightarrow \mathbb{R}^{n \times n}$ is in $L_{n \times n}^1$, the sequence $\{t_i\}_0^{\infty}$ is in $[0, \infty)$ with $t_0 = 0$, $t_i > 0$ for $i \geq 1$ and $\{A_i\}_{i=0}^{\infty}$ is a sequence of matrices in $\mathbb{R}^{n \times n}$

subject to $\sum_{i=0}^{\infty} |A_i| < \infty$ and δ is the Dirac "function." The set of elements in $\mathcal{A}^{n \times n}$ constitute a non-commutative Banach algebra with a unit, with the usual definition for addition, the product defined by convolution, and the

norm defined by

$$\|A\|_a \triangleq \int_0^{\infty} |A_a(t)| dt + \sum_{i=0}^{\infty} |A_i|.$$

These facts are well-known [6,15].

The symbol " $\hat{\cdot}$ " over a function, such as \hat{f} , denotes the Laplace transform of f : it is defined by

$$\hat{f}(s) \triangleq \int_0^{\infty} f(t) e^{-st} dt.$$

For distributions, it is defined according to L. Schwartz [13] or, by using Stieltjes integrals, according to Widder [14]. The subscript T , as in f_T , denotes the truncation of the function f at time T , namely

$$f_T(t) = \begin{cases} f(t) & \text{for } 0 \leq t \leq T \\ 0 & \text{for } t > T \end{cases}$$

Finally $\hat{\mathcal{A}}^{n \times n}$ denotes the algebra of Laplace transforms of elements in $\mathcal{A}^{n \times n}$ (with pointwise product).

3. System Description and Assumptions.

We consider a $2n$ -input $2n$ -output nonlinear time-varying feedback system S as shown in Fig. 1. The inputs u_1, u_2 , errors e_1, e_2 , outputs y_1, y_2 are functions of time mapping \mathbb{R}_+ into \mathbb{R}^n . The block labeled ϕ is a memoryless, time-varying nonlinearity whose input-output relation is defined in terms of a nonlinear function $\phi: \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ by

$$y_2(t) = \phi[e_2(t), t]. \quad (1)$$

The nonlinear function $\phi(\cdot, \cdot)$ satisfies the following assumptions:

- (N.1) $\phi(\cdot, \cdot): \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ and ϕ is a continuous function with respect to its first argument and is a regulated function⁽⁺⁾ with respect to its
- (+) $\phi(x, t): \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is called regulated in t iff for all fixed $x \in \mathbb{R}^n$, $t \mapsto \phi(x, t)$ has finite one-sided limits at every $t \in \mathbb{R}_+$.

second argument.

(N.2) There exists a nonsingular matrix $F \in \mathbb{R}^{n \times n}$ and a positive real number μ such that

$$|\phi(x, t) - \phi(x', t) - F(x - x')| \leq \mu |x - x'| \quad (2)$$

for all $t \in \mathbb{R}_+$ and all $x, x' \in \mathbb{R}^n$; moreover

$$\phi(0, t) = 0 \quad \text{for all } t \in \mathbb{R}_+. \quad (3)$$

The block labeled G is a linear time-invariant subsystem whose input-output relation is defined in terms of its impulse response matrix G by convolution, i.e.

$$y_1(t) = (G * e_1)(t) \quad \text{for all } t \in \mathbb{R}_+ \quad (4)$$

G is a matrix valued distribution on $[0, \infty)$ whose Laplace transform \hat{G} satisfies the assumption (G):

$$\hat{G}(s) = \sum_{k=1}^{\ell} \sum_{\alpha=0}^{m_k-1} R_{k\alpha} (s - p_k)^{-m_k+\alpha} + \hat{G}_\rho(s) \quad \text{for } \operatorname{Re} s \geq 0, \quad (5)$$

where $\operatorname{Re} p_k \geq 0$ for $k = 1, 2, \dots, \ell$; the poles p_k and the coefficient matrices $R_{k\alpha}$ are either real or occur in complex conjugate pairs;

$\hat{G}_\rho(s) \in \hat{Q}^{n \times n}$. The system equations are (1), (4) and

$$e_1 = u_1 - y_2 \quad (6)$$

$$e_2 = u_2 + y_1. \quad (7)$$

Definition: Let $p \in [1, \infty]$; the system S (Fig. 1) defined by (1) - (7) is said to be L^p -stable iff the maps $(u_1, u_2) \mapsto (e_1, e_2)$ and $(u_1, u_2) \mapsto (y_1, y_2)$ are L^p -stable i.e. to any input pair (u_1, u_2) belonging to L_{2n}^p corresponds an error pair (e_1, e_2) and an output pair (y_1, y_2) both belonging to L_{2n}^p and there is a number $k \in \mathbb{R}_+$ such that

$$\|e_1\|_p + \|e_2\|_p \leq k [\|u_1\|_p + \|u_2\|_p]$$

$$\|y_1\|_p + \|y_2\|_p \leq k [\|u_1\|_p + \|u_2\|_p]$$

for all $(u_1, u_2) \in L_{2n}^p$.

4. Main Result.

Theorem 1. Consider the system S described by (1), (4), (6) and (7), where the assumptions (G), (N.1) and (N.2) are satisfied. Let H_F be the closed-loop impulse response of the n -input n -output convolution feedback system $u_1 \mapsto y_1$ with G as open-loop impulse response and F as constant feedback matrix, i.e.

$$\hat{H}_F = \hat{G}[I + F\hat{G}]^{-1}. \quad (8)$$

In (5) for $k = 1, 2, \dots, \ell$ set

$$R_k(s) \triangleq \sum_{\alpha=0}^{m_k-1} R_{k\alpha}(s-p_k)^{-m_k+\alpha}. \quad (9)$$

At each pole p_k for $k = 1, 2, \dots, \ell$ consider the Laurent expansion of $I + F\hat{G}(s)$ up to and including the constant term. This proper rational function can be represented as the product $N_k(s) D_k(s)^{-1}$ where N_k and D_k are right-coprime polynomial matrices [18-21], i.e. for $k = 1, 2, \dots, \ell$

$$N_k(s) D_k(s)^{-1} = I + F[R_k(s) + \sum_{\substack{\beta=1 \\ \beta \neq k}}^{\ell} R_{\beta}(p_k) + \hat{G}_{\rho}(p_k)]. \quad (10)$$

Under these conditions, if

$$\inf_{\operatorname{Re} s \geq 0} |\det[I + F\hat{G}(s)]| > 0 \quad (11)$$

$$\det N_k(p_k) \neq 0 \quad \text{for } k = 1, 2, \dots, \ell \quad (12)$$

and

$$\|H_F\|_a^p < 1 \quad (13)$$

then,

- (i) for any $p \in [1, \infty]$, the maps $(u_1, u_2) \mapsto (e_1, e_2)$ and $(u_1, u_2) \mapsto (y_1, y_2)$ are well-defined maps sending L_{2n}^p into L_{2n}^p ;
- (ii) for any $p \in [1, \infty]$, these maps are uniformly continuous on L_{2n}^p ;
- (iii) for any $p \in [1, \infty]$, the system S is L^p -stable.

5. Proof. To prove Theorem 1, we need two lemmas.

Lemma 1. Consider a special case of the system S (Fig. 1), where for all $e_2 \in \mathbb{R}^n$, all $t \in \mathbb{R}_+$, $\phi(e_2, t) = Fe_2$, with F a nonsingular element of $\mathbb{R}^{n \times n}$. Let the open-loop transfer function matrix \hat{G} be defined by

- (5). Let N_k and D_k be the right-coprime polynomial matrices defined by
- (10). Under these conditions

$$[I + F\hat{G}]^{-1} \in \hat{A}^{n \times n}$$

and

$$\hat{H}_F \triangleq \hat{G}[I + F\hat{G}]^{-1} \in \hat{A}^{n \times n}$$

if and only if

$$\inf_{\operatorname{Re} s \geq 0} |\det[I + F\hat{G}(s)]| > 0 \quad (11)$$

and

$$\det N_k(p_k) \neq 0 \quad \text{for } k = 1, 2, \dots, \ell. \quad (12)$$

This is a generalized version of a result of [10].

Lemma 2. Consider a more general system than the one shown in Fig. 1, in that \bar{G} and ϕ are replaced by \bar{H}_1 and \bar{H}_2 respectively. Let p be fixed and $p \in [1, \infty]$. Let \bar{H}_1 and \bar{H}_2 be nonanticipative maps of L_{ne}^p into L_{ne}^p . Let \bar{H}_1 be linear, thus $\bar{H}_1 0 = 0$. Let $\bar{H}_2 0 = 0$. Let e_1, e_2 and u_1, u_2 be defined by the system equations. Under these conditions if

(a) for some $F \in \mathbb{R}^{n \times n}$, F nonsingular, $(I + FH_1)^{-1}$ maps L_{ne}^p into L_{ne}^p and is nonanticipative;

(b) there exists some positive real number μ such that

$$\|(H_2 e_2)_T - (H_2 e'_2)_T - F(e_2 - e'_2)_T\|_p \leq \mu \|e_{2T} - e'_{2T}\|_p$$

for all $T \in [0, \infty)$ and for all $e_2, e'_2 \in L_{ne}^p$;

(c) $\|H_1(I + FH_1)^{-1}\| < \infty$;

(d) $\gamma = \|H_1(I + FH_1)^{-1}\|_\mu < 1$,

then: (i) given any input pair (u_1, u_2) in L_{2ne}^p , a unique error e_2 in L_{ne}^p is obtained by a fixed point iteration starting from an arbitrary point;

(ii) if u_1 and u_2 are the zero elements in L_{ne}^p , then e_2 is the zero element in L_{ne}^p ;

(iii) to any two input pairs, say $(u_1, u_2), (u'_1, u'_2)$ in L_{2ne}^p , there correspond two errors e_2 and e'_2 in L_{ne}^p such that

$$\|e_{2T} - e'_{2T}\|_p \leq (1-\gamma)^{-1} \|F^{-1}(I + FH_1)^{-1} F(u_{2T} - u'_{2T})\|_p +$$

$$\|H_1(I + FH_1)^{-1}(u_{1T} - u'_{1T})\|_p \quad \forall T \in [0, \infty).$$

Therefore the map $(u_1, u_2) \mapsto e_2$ is a well-defined L^p -stable map sending L_{2n}^p into L_n^p which is uniformly continuous on L_{2n}^p .

This Lemma is a consequence of the loop transformation technique [12] and the fixed point Theorem [16].

Proof of Theorem 1. Let F be the nonsingular $n \times n$ constant matrix of assumption (N.2). Make the system transformation such that the block in the forward path becomes

$$H_F = G(I + FG)^{-1} \tag{14}$$

and the block in the feedback path becomes

$$\psi = \phi - FI. \quad (15)$$

Let $\hat{H}_F(s)$ be the transfer function matrix of H_F , then

$$\hat{H}_F(s) = \hat{G}(s)(I + F\hat{G}(s))^{-1}. \quad (16)$$

By assumptions (11) and (12) of Theorem 1, Lemma 1 implies that $(I + F\hat{G}(s))^{-1}$ and $\hat{H}_F(s)$ are in $\hat{Q}^{n \times n}$; since they are the transfer functions of the operators $(I + FG)^{-1}$ and H_F , these operators are nonanticipative, send L_n^p into L_n^p for any $p \in [1, \infty)$, and are L^p -stable for all $p \in [1, \infty]$, [6]. Thus the impulse response matrix H_F is in $Q^{n \times n}$ and is of the form

$$H_F(t) = \begin{cases} H_a(t) + \sum_{i=0}^{\infty} H_i \delta(t - t_i) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0, \end{cases}$$

where $H_a \in L_{n \times n}^1$, the H_i 's are constant matrices such that $\sum_{i=0}^{\infty} |H_i| < \infty$ and $t_0 = 0$, $t_i > 0$ for $i \geq 1$. Also H_F has a well-defined norm in $Q^{n \times n}$

$$\|H_F\|_a \triangleq \int_0^{\infty} |H_a(t)| dt + \sum_{i=0}^{\infty} |H_i|.$$

Note that $\|H_F\|_a$ is the induced operator norm when $p = \infty$ and is an upper bound on the induced operator norm when $p \neq \infty$. By assumption (N.2) we have

$$\|(\phi e_2)_T - (\phi e'_2)_T - F(e_2 - e'_2)_T\|_p \leq \mu \|e_2 - e'_2\|_p$$

for all $T \in [0, \infty)$, for all $e_2, e'_2 \in L_{ne}^p$.

Finally by assumption (13): $\|H_F\|_a \mu < 1$; furthermore G is linear so $G 0 = 0$ and, by assumption (N.2), $\phi 0 = 0$. So all the conditions of Lemma 2 are met for any $p \in [1, \infty]$ with $H_1 = G$ and $H_2 = \phi$. Hence, for any

$p \in [1, \infty]$, it follows that for the system S the map $(u_1, u_2) \rightarrow e_2$ is well defined sending L_{2n}^p into L_n^p , is L^p -stable and is uniformly continuous on L_{2n}^p . Since $y_2 = \phi e_2$,

$$\|(\phi e_2)_T - (\phi e'_2)_T\|_p = \|F(e_{2T} - e'_{2T})\|_p \leq$$

$$\|(\phi e_2)_T - (\phi e'_2)_T - F(e_{2T} - e'_{2T})\|_p \leq \mu \|e_{2T} - e'_{2T}\|_p,$$

and $\phi 0 = 0$,

it follows for any $p \in [1, \infty]$, that the map $e_2 \mapsto y_2$ is a well-defined map sending L_n^p into L_n^p which is L^p -stable and uniformly continuous on L_n^p .

Finally since $e_1 = u_1 - y_2$ and $y_1 = e_2 - u_2$, the conclusion of the theorem follows.

6. Final Remark. If, instead of assuming that ϕ satisfies an incremental gain condition as in (2) of assumption (N.2), we had assumed that there exists a positive real number μ such that

$$|\phi(x, t) - Fx| \leq \mu |x| \quad \text{for all } t \in \mathbb{R}_+, \quad \text{for all } x \in \mathbb{R}^n, \quad (2')$$

then we would be able to use the small gain theorem to prove the following: suppose that for some $p \in [1, \infty]$ and for any input pair $(u_1, u_2) \in L_{2n}^p$ the error pair $(e_1, e_2) \in L_{2ne}^p$, then assumptions (N.1), (2'), (3), (G) and (11), (12), (13) imply that system S is L^p -stable. This result is easily obtained by standard techniques [1], [2], [11] and extends a recent result of Prada and Bickart [17]. Note that under the relaxed assumption (2') we do not guarantee existence, nor uniqueness, nor continuous dependence.

7. Conclusion. We have shown that if the given nonlinear time-varying feedback system S will be uniquely defined, stable and continuously dependent on its inputs in any L^p norm if eventually i) the unstable convolution subsystem can be stabilized by a constant feedback gain F and ii) if the incremental gain of the difference of the nonlinear gain function ϕ and F is sufficiently small.

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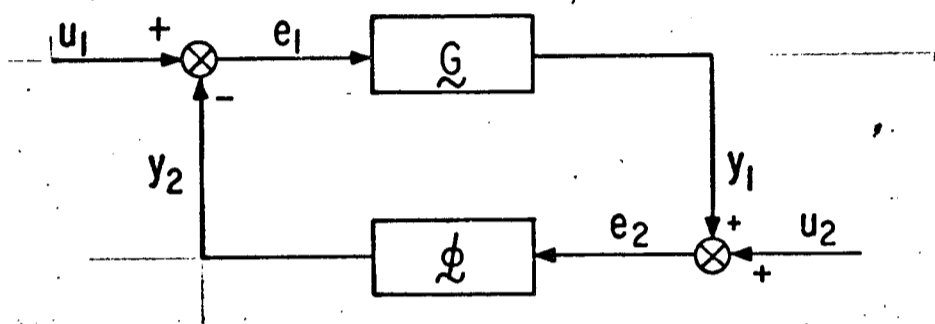


Fig. 1. The system S.